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Better Expanders and Superconcentrators

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Explicit construction of families of linear expanders and superconcentrators is relevant to theoretical computer science in several ways. Here we construct better expanders than those previously known and use them to construct explicitly n-superconcentrators with $\approx 122.74n$ edges; much less than the previous most economical construction. © 1987 Academic Press, Inc.

1. INTRODUCTION

An (n, k, c)-*expander* is a k-regular bipartite graph on the sets of vertices I (inputs) and O (outputs), where $|I| = |O| = n$, and every set of $x \le n/2$ inputs is joined by edges to at least $x + c(1 - x/n)x$ different outputs. A family of linear expanders of *density* k and *expansion* c is a set $\{G_i\}_{i=1}^{\infty}$, where G_i is an (n_i, k, c)-expander, $n_i \to \infty$ and $n_{i+1}/n_i \to 1$ as $i \to \infty$.

Such a family is the main component in the recent parallel sorting network of Ajtai *et al.* It also forms the basic building block used in the construction of graphs with special connectivity properties and small num-

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ber of edges (see, e.g., Chung [Ch]). An example of a graph of this type is an *n-superconcentrator* (s.c.), which is a directed acyclic graph with n inputs and n outputs such that for every $1 \leq r \leq n$ and every two sets A of r inputs and B of r outputs there are r vertex disjoint paths from the vertices of A to the vertices of B. A family of linear s.c.-s of *density* k is a set $\{G_n\}_{n=1}^{\infty}$, where G_n is an n-s.c. with $\leq (k + o(1))n$ edges. Superconcentrators, which are the subject of an extensive literature, are relevant to computer science in several ways. They have been used in the construction of graphs that are hard to pebble (see Lengauer and Tarjan [LT], Pippenger [P2] and Paul *et al.* [PTC]), in the study of lower bounds (see Valiant [Va]) and in the establishment of time space tradeoffs for computing various functions (Abelson [Ab], Ja'Ja [Ja], and Tompa [To]).

It is not too difficult to prove the existence of a family of linear expanders (and hence a family of linear s.c.-s) using probabilistic arguments (see, e.g., Pinsker [Pi], Pippenger [P1], and Chung [Ch]). However, for applications an explicit construction is desirable. Such a construction is much more difficult and the search for explicit, economic linear expanders and s.c.-s attracted considerable attention. Margulis [Ma] was the first to construct a family of linear expanders. A similar family is used in [GaGa] to construct n-s.c.-s with $\approx 271.8n$ edges. This was improved by Chung to $\approx 261.5n$ and later, by Buck [Bu] to $\approx 190n$, and by Alon and Milman [AM1, AM2] to $175n$.

Very recently Jimbo and Maruoka [JM] constructed slightly different expanders than those of [Ma, GaGa] that enabled them to produce s.c.-s of density 248. In this note we modify their construction and obtain a family of expanders that supplies s.c.-s of density ≈ 122.74.

Shamir [Sh] constructed families of nonacyclic directed s.c.-s of density ≈ 204 and of undirected s.c.-s of density ≈ 118. Our new expanders enable us to improve these densities to ≈ 49.84 and ≈ 30.3, respectively.

Our results are proved by combining several results from [GaGa] and [JM] with some of the ideas of [AM1, AM2] and [A1] about the connection between the eigenvalues of the adjacency matrix of a graph and its expansion properties. Our paper is organized as follows. In Section 2 we construct our expanders and estimate their expansion properties. In Section 3 we use our expanders to construct better s.c.-s. In Section 4 we construct better nonacyclic and undirected s.c.-s. Section 5 contains some open problems.

2. Better Expanders

Our expanders are double covers of nonbipartite graphs. Let G be a graph on the set of vertices $\{v_1, v_2, \ldots, v_n\}$. The (extended) *double cover*

$\bar{G} = (I, O, E)$ of G is a bipartite graph on the classes of vertices $I = \{x_1, x_2, \dots, x_n\}$ and $O = \{y_1, y_2, \dots, y_n\}$, in which $x_i y_i \in E$ for $1 \le i \le n$ and $x_i y_j \in E$ iff $v_i v_j$ is an edge of G. For a graph $G = (V, E)$ and for $X \subseteq V$ put $N_G(X) = \{y \in V: xy \in E$ for some $y \in X\}$. Clearly if $\bar{G} = (I, O, E)$ is the double cover of G, X is a set of vertices of G and $\bar{X} \subseteq I$ is the corresponding set of inputs in $\bar{G}$, then $|N_{\bar{G}}(X)| = |X \cup N_G(X)| = |X| + |N_G(X) - X|$. Hence in order to estimate the expansion properties of $\bar{G}$ one has to estimate the quantities $|N_G(X) - X|$.

Let $n = m^2$ and let A_m be $\{0, 1, \dots, m-1\} \times \{0, 1, \dots, m-1\}$. Define the following 6 permutations on A_m,

$$\sigma_1(x, y) = (x, y + 2x), \qquad \sigma_2(x, y) = (x, y + 2x + 1),$$
$$\sigma_3(x, y) = (x, y + 2x + 2) \qquad \sigma_4(x, y) = (x + 2y, y),$$
$$\sigma_5(x, y) = (x + 2y + 1, y), \qquad \sigma_6(x, y) = (x + 2y + 2, y),$$

where all additions are modulo m.

Let T_n denote the 8-regular graph on the vertex set $V = A_m$ in which $(x, y) \in V$ is joined by edges to $\sigma_i(x, y)$ (and to $\sigma_i^{-1}(x, y)$) for $i = 1, 2, 4, 5$. Let H_n denote the 12-regular graph on the vertex set $V' = A_m$ in which $(x, y) \in V'$ is joined by edges to $\sigma_i(x, y)$ (and to $\sigma_i^{-1}(x, y)$) for $1 \le i \le 6$. Let $\bar{T}_n$ and $\bar{H}_n$ denote the double covers of T_n and H_n, respectively. For any set $A \subseteq V$ put $a = |A|/|V| = |A|/n$. As shown in Corollary 2.3 below, the family $\bar{T}_n$ has remarkable expansion properties, which will be used later for constructing efficient s.c.-s. The expansion properties of the family $\bar{H}_n$ can be approximated similarly (see Remark 2.4), but since the best constants are obtained from $\bar{T}_n$, we deal here mainly with this family.

Our main tool for estimating the expansion properties of a graph is the following theorem. More details on the tight connection between λ_G defined below, and the expansion properties of a graph G appear in [A1].

THEOREM 2.1. *Let $G = (V, E)$ be a graph with maximal degree ρ. Let $Q = Q_G = (q_{uv})_{u, v \in V}$ be the matrix given by $q_{uu} = \rho(u)$ for all $u \in V$, where $\rho(u)$ is the degree of u, $q_{uv} = -1$ for $u \ne v$, $uv \in E$ and $q_{uv} = 0$ for $u \ne v$, $uv \notin E$. Let $\lambda = \lambda_G \ge 0$ be the second smallest eigenvalue of Q. If $X \subseteq V$, $W = N_G(X) - X$ then*

$$w^2 - 2(1 - 2x - \alpha)w - 4x(1 - x) \ge 0,$$
$$\text{where } \alpha = (1 + 2d)/4d,\ d = \lambda/2\rho. \quad (2.1)$$

Proof. Put $|V| = n$. By Rayleigh's principle, if $f: V \to \Re$ is a function and $\Sigma_{v \in V} f(v) = 0$ (i.e., f is orthogonal to the eigenvector of 0 = the smallest eigenvalue of Q) then

$$(Qf, f) = \sum_{uv \in E} (f(u) - f(v))^2 \geq \lambda \sum_{v \in V} f^2(v). \tag{2.2}$$

Put $Y = V - (X \cup W)$. If $x = 0$ or $y = 0$ (2.1) is trivial. Otherwise define a function φ: $V \to \Re$ by $\varphi(v) = 1/x$ for $v \in X$, $\varphi(v) = -1/y$ for $v \in Y$ and $\varphi(v) = \frac{1}{2}((1/x) - (1/y))$ for $v \in W$. Define $f = \varphi - b$, where $b = (1/n)\Sigma_{v \in V}\varphi(v) = \frac{1}{2}((1/x) - (1/y))w$. Clearly $\Sigma_{v \in V} f(v) = 0$ and hence it satisfies (2.2). However, if $uv \in E$ then $|f(u) - f(v)| = \frac{1}{2}((1/x) + (1/y))$ if $|\{u, v\} \cap W| = 1$ and 0 otherwise. Therefore,

$$\sum_{uv \in E} (f(u) - f(v))^2 \leq \sum_{v \in W} \rho(v)\frac{1}{4}\left(\frac{1}{x} + \frac{1}{y}\right)^2 \leq \rho \cdot n \cdot w \cdot \frac{1}{4}\left(\frac{1}{x} + \frac{1}{y}\right)^2.$$

Substituting in (2.2) we thus obtain

$$\begin{aligned}\rho \cdot n \cdot w \cdot \frac{1}{4}\left(\frac{1}{x} + \frac{1}{y}\right)^2 &\geq \lambda \sum_{v \in V} f^2(v)\\ &= \lambda\left[\left(\frac{1}{x} - b\right)^2 nx + \left(-\frac{1}{y} - b\right)^2 ny\right.\\ &\qquad \left. + \left(\frac{1}{2}\left(\frac{1}{x} - \frac{1}{y}\right) - b\right)^2 nw\right],\end{aligned}$$

or, since $4\lambda/\rho = 8d$,

$$w\left(\frac{1}{x} + \frac{1}{y}\right)^2 \geq 8d\left[\left(\frac{1}{x} - b\right)^2 x + \left(-\frac{1}{y} - b\right)^2 y + \left(\frac{1}{2}\left(\frac{1}{x} - \frac{1}{y}\right) - b\right)^2 w\right].$$

Inequality (2.1) follows from the last inequality by simple arithmetic manipulations. These appear in Appendix 1. □

Combining Theorem 2.1 with the results of [JM] we obtain

THEOREM 2.2. *Let X be a set of vertices of $T = T_n$. Put $W = N_T(X) - X$. Then*

$$w^2 - 2(1 - 2x - \alpha)w - 4x(1 - x) \geq 0, \tag{2.3}$$

where

$$\alpha = (1 + 2d)/4d, \ d = (8 - 5\sqrt{2})/16. \tag{2.4}$$

Proof. Jimbo and Maruoka [JM] proved that $\lambda = \lambda_T \geq 8 - 5\sqrt{2}$. Clearly here $\rho = 8$. The result now follows from Theorem 2.1. □

Let w_1, w_2 $(w_1 < w_2)$ be the two roots of (2.3). Clearly (2.3) implies $w \geq w_2 = -4x(1-x)/w_1$. Substituting the value of w_1 we obtain the following corollary.

COROLLARY 2.3. Let T, X, W, x, w and α be as in Theorem 2.2. Then $w \geq 4x(1-x)/(-1 + 2x + \alpha + \sqrt{(1-\alpha)^2 + 4x\alpha})$. Thus for $x \leq \frac{1}{2}$, $w \geq 4x(1-x)/(\alpha + \sqrt{1+\alpha^2})$ and hence $\overline{T}_n$ is an $(n, 9, c)$-expander, where $c = 4/(\alpha + \sqrt{1+\alpha^2}) \approx 0.412$.

Remark 2.4. Combining Theorem 2.1 with the results of [JM] (or with some of the results of [GaGa]), we can obtain similar estimates for the expansion properties of the graph H_n defined above. In particular we can show that $\overline{H}_n$ is an $(n, 13, c')$-expander, where $c' \approx 0.466$. We omit the details.

3. BETTER SUPERCONCENTRATORS

Theorem 3 in [GaGa] shows how to construct from a family of linear expanders of density k and expansion $2/(p-1)$ a family of linear s.c.-s of density $(2k+3)p+1$. By corollary 2.3, the expansion of $\overline{T}_n$ is $\approx 0.412 > \frac{2}{5}$, which supplies s.c.-s of density $(2 \cdot 9 + 3) \cdot 6 + 1) = 127$.

The method of Appendix 1 of [GaGa] enables us to improve these constructions to ≈ 122.74, as shown below. It is worth noting that the method used by Chung to improve the construction of [GaGa] does not apply here, since it can work only if the expansion is very close to a number of the form $2/(p-1)$, where p is an integer.

An (n, θ, k) *bounded concentrator* (b.c.) is a bipartite graph with n inputs, θn outputs, $(\frac{1}{2} < \theta < 1)$, and kn edges such that for every subset X of inputs of size at most $n/2$, $|N(X)| \geq |X|$. In [P1] and in [GaGa] it is shown how to construct linear s.c.-s from a family of b.c.-s. Thus, e.g., our n-s.c.-s of density 127 mentioned above are constructed from a family of $(n, 6/7, k)$ b.c.-s, which are produced using the expanders $\overline{T}_{6/7n}$. We now show how to construct from the $\overline{T}_m - s$ (n, θ, k) b.c.-s with $\frac{5}{6} < \theta < \frac{6}{7}$ and $k = k(\theta)$. Let $\frac{5}{6} < \theta < \frac{6}{7}$ be a parameter, to be determined later, with θn a square integer. We construct a b.c. $B = B_n$ from the expander $\overline{T}_{\theta n}$ as follows. The set O of θn outputs of B is the set of outputs of $\overline{T}_{\theta n}$. The set I of n inputs of B is $S \cup R$, where S is the set of inputs of $\overline{T}_{\theta n}$ and R is a set of $(1-\theta)n$ additional vertices. The set of edges of B consists of the edges of $T_{\theta n}$ (from S to O) together with θn edges which connect R to O in a one to one fashion: the ith vertex of R is joined to the jth vertex of O iff

$i \equiv j \bmod((1-\theta)n)$. Clearly B has $10\theta n$ edges. Suppose $X \subset I$ satisfies $x \equiv |X|/n \le \frac{1}{2}$. Put $X_1 = X \cap R$, $X_2 = X \cap S$, $Y_i = N(X_i)$, $x_i = |X_i|/n$ and $y_i = |Y_i|/n$ $(i = 1, 2)$. θ is chosen such that either $y_1 \ge x$ or $y_2 \ge x$ (and thus B is a b.c.). Since $(6 - 7\theta)n$ is the number of inputs in R that are connected to only 5 outputs we have

$$y_1 \ge \begin{cases} 5x_1, & x_1 \le 6 - 7\theta \\ 6x_1 - (6 - 7\theta), & x_1 > 6 - 7\theta. \end{cases}$$

If $y_1 \ge x$ we are done. Otherwise $y_1 \le x$, i.e.,

$$x_1 \le \begin{cases} \dfrac{x}{5} & \text{if } x \le 5(6 - 7\theta) \\ \dfrac{x + (6 - 7\theta)}{6} & \text{otherwise.} \end{cases}$$

Thus

$$x_2 \ge \begin{cases} \dfrac{4x}{5} & \text{if } x \le 5(6 - 7\theta) \\ \dfrac{5x - (6 - 7\theta)}{6} & \text{otherwise.} \end{cases}$$

In the worst case equality holds. For $y_2 \ge x$ to hold, a set of $x_2 n$ inputs in $\overline{T}_{\theta n}$ must be connected to at least xn outputs. In Appendix 2 we substitute the various parameters into (2.3) to obtain the best possible (i.e., smallest possible) $\theta = 0.85288$. In Lemma 8 of [GaGa] it is shown (using the construction of [P1]) how a sequence of (n, θ, k) b.c.-s supply n-s.c.-s of density $(2k + 1)/(1 - \theta)$. In our case $k = 10\theta$ and the density of the s.c.-s is $(20\theta + 1)/(1 - \theta) \approx 122.74$.

Similarly, if we use $\overline{H}_n$ instead of $\overline{T}_n$ we obtain s.c.-s of density ≈ 157.35.

4. Weaker s.c.-s

Shamir [Sh] constructed two types of weaker s.c.-s: nonacyclic s.c.-s and undirected s.c.-s. Here we improve the density of both his constructions using our expanders.

We first consider nonacyclic s.c.-s. Let $G = (V, E)$ be a directed graph, where V is the disjoint union of 3 sets; I (inputs), O (outputs) and N. Here $|I| = |O| = n$ and $|N| = 5n$. The set of edges $E \subseteq (I \cup N) \times (N \cup O)$

consists of three parts: (1) The edges of a graph of the form T_{5n} with every edge taken in both directions on the set of vertices N. The number of these edges is $5n \cdot 8 = 40n$. (2) A matching of n edges from I to O. (3) $5n$ edges from N to O joined in a one to one fashion (i.e., each vertex of N is joined to precisely one input and one output). Altogether we have $51n$ edges. We proceed to show that G is an n-s.c. Otherwise, as is shown in [Sh], Menger's Theorem implies that there are $A \subseteq I$, $B \subseteq O$, with $|A| = |B| \le n/2$ and $F \subseteq N$, $|F| < |A|$, such that F separates A and B. Without loss of generality we may assume that the set $A' \subseteq N - F$ of vertices reachable from A by paths that avoid F satisfies $|A'| \le |N - F|/2 \le 5n/2$. (Otherwise replace A and B.) Let $A'' = N_G(A') \cap N$. Clearly $|A'| \ge 5|A| - |F| \ge 4|A|$. We will show that

$$|A'' - A'| \ge |A| > |F|. \tag{4.1}$$

This implies that $A'' - (A' \cup F) \neq 0$, contradicting the definition of A'. To prove (4.1) put $a = |A|/|N| = |A|/5n$ $(0 \le a \le \frac{1}{10}$, $x = |A'|/5n$ $(4a \le x \le \frac{1}{2})$ and $w = |A'' - A'|/5n$ $(w \ge 0)$. By Theorem 2.1,

$$f(w) = w^2 - 2(1 - 2x - \alpha)w - 4x(1 - x) \ge 0,$$

where α is as in (2.4). We must show that this implies $w \ge a$. It is enough to check that for each permissible values of x, a the value of $f(w)$ for $w = a$ is at most 0, i.e., that for $4a \le x \le \frac{1}{2}$, $a^2 - 2(1 - 2x - \alpha)a - 4x(1 - x) \le 0$. It is enough to check this for $x = 4a$ and $x = \frac{1}{2}$ (for each $0 \le a \le \frac{1}{10}$). For $x = \frac{1}{2}$ we have to check that $a^2 + 2\alpha a - 1 \le 0$. For $x = 4a$ we have to check that $a^2 - 2(1 - 8a - \alpha)a - 16a(1 - 4a) \le 0$, i.e., $a \le (18 - 2\alpha)/81$. Substituting the value of α from (2.4) it follows that both inequalities hold for $0 \le a \le \frac{1}{10}$. This proves (4.1) and shows that G is an acyclic s.c., as needed.

By ignoring the direction of edges of G we obtain an undirected n-s.c. with $(8 \cdot 5n/2 + 11n = 31n$ edges. One can slightly improve these densities using the method discussed before for acyclic s.c.-s and obtain nonacyclic s.c.-s with $49.84n$ edges and undirected s.c.-s with $30.3n$ edges. For details see Appendix 3.

5. Open Problems

(1) Bassalygo [Ba] proved the existence of a family of (acyclic) s.c.-s of density ≈ 36. One might try to improve our explicit construction and obtain an explicit family with a similar density.

(2) A very intriguing problem, mentioned in [GaGa], is to find a purely combinatorial proof that a given explicit family of graphs is a family

of linear expanders. All the proofs known at present [Ma, GaGa, AM1, JM, Sc, Bu] are not elementary.

Appendix 1

We obtained the inequality

(1) $A \geq B$, where

$$A = w\left(\frac{1}{x} + \frac{1}{y}\right)^2, \quad B = 8d\left[\left(\frac{1}{x} - b\right)^2 x + \left(-\frac{1}{y} - b\right)^2 y + \left(\frac{1}{2}\left(\frac{1}{x} - \frac{1}{y}\right) - b\right)^2 w\right],$$

$$b = \frac{1}{2}\left(\frac{1}{x} - \frac{1}{y}\right)w \quad \text{and} \quad x + y + w = 1.$$

So,

$$\begin{aligned}
B(xy)^2/2d &= 4(xy)^2\left[\frac{1}{x} + \frac{1}{y} + b^2(x+y) + \frac{1}{4}(w-1)^2\left(\frac{1}{x} - \frac{1}{y}\right)^2 w\right] \\
&= 4xy(x+y) + (x-y)^2 w^2 (x+y) + (x+y)^2(x-y)^2 w \\
&= (x+y)\left[4xy + (x-y)^2 w(w+x+y)\right] \\
&= (x+y)\left[4xy + (x-y)^2(1-(x+y))\right] \\
&= (x+y)\left[(x+y)^2 - (x-y)^2(x+y)\right] \\
&= (x+y)^2\left[(x+y) - (x-y)^2\right].
\end{aligned}$$

Substituting into (1) we get

$$w \geq 2d\left[1 - w - (w + 2x - 1)^2\right]$$

or

$$2\alpha w = w\left(1 + \frac{1}{2d}\right) \geq 1 - (w + 2x - 1)^2$$

from which (2.1) follows.

APPENDIX 2

We have

$$x_2 = \begin{cases} \dfrac{4x}{5} & \text{if } x \le 5(6 - 7\theta) \\ \dfrac{5x - (6 - 7\theta)}{6} & \text{otherwise.} \end{cases}$$

Recall (2.1) $\hat{w}^2 - 2(1 - 2\hat{x} - \alpha)\hat{w} - 4\hat{x}(1 - \hat{x}) \ge 0$. (We rename the variables to avoid confusion with $x = |X|/n$). In our case $\hat{x} = x_2/\theta$ and $\hat{w} \ge x_1/\theta$ (since $y_2 \ge x$). To guarantee $\hat{w} \ge x_1/\theta$ we must have

$$\left(\frac{x_1}{\theta}\right)^2 - 2\left(1 - 2\frac{x_2}{\theta} - \alpha\right)\frac{x_1}{\theta} - 4\left(\frac{x_2}{\theta}\right)\left(1 - \left(\frac{x_2}{\theta}\right)\right) \le 0. \quad (1)$$

We now consider two cases:

Case 1. $x \le 5(6 - 7\theta)$ *and* $x_2 = 4x/5 = 4x_1$.

Substituting into (1) we get $(x_1/\theta) - 2(1 - (8x_1/\theta) - \alpha) - 16(1 - (4x_1/\theta)) \le 0$ or $81x_1 \le (18 - 2\alpha)\theta$. But $x_1 = x/5 \le 6 - 7\theta$. So the required inequality holds if $81(6 - 7\theta) \le (18 - 2\alpha)\theta$. Taking $\beta = (18 - 2\alpha)/81$, we get $\theta \ge 6/(7 + \beta) = 0.84465$.

Case 2. $5(6 - 7\theta) \le x \le 1/2$.

In this case, $x_2 = (5x - (6 - 7\theta))/6$ and $x_1 = (x + (6 - 7\theta))/6$ are substituted into (1). For a fixed θ we get a quadratic in x, and for (1) to hold it must hold for $x = 5(6 - 7\theta)$ and for $x = \frac{1}{2}$. The former is covered by Case 1 (for $\theta \ge 0.84465$) so we need only consider $x = \frac{1}{2}$ and substitute $x_1 = (6.5 - 7\theta)/6$ and $x_2 = (-3.5 + 7\theta)/6$ in (1). We obtain $\theta^2(77 - 7\gamma) + \theta(6.5\gamma - 105) + 0.25 \le 0$ where $\gamma = 16 + 12\alpha$, which yields $\theta \ge 0.85288$. Choosing $\theta = 0.85288$ guarantees that (1) holds in both cases.

APPENDIX 3

We improve both constructions by taking $|N| = tn$, with $t = 121/(44 - 4\alpha) \approx 4.884$. As a result, $(5 - t)n$ elements of I and O will be connected to only 4 elements of N. The density of the nonacyclic (undirected) s.c.-s is $10t + 1 \approx 49.84$ $(6t + 1 \approx 30.3)$.

Let A, A', A'' and F be as in Section 4. We chose the smallest t such that (4.1) still holds (and thus the resulting graph is the corresponding s.c.).

To prove (4.1) put $a = |A|/|N| = |A|/tn$, $x = |A'|/tn$

$$1/2 \geq x \geq \begin{cases} 3a & \text{if } a \leq \dfrac{5-t}{t} \\ 4a - \dfrac{(5-t)}{t} & \text{otherwise,} \end{cases}$$

and $w = |A'' - A'|/tn$ ($w \geq 0$). By Theorem 2.2

$$g(x, w) \equiv w^2 - 2(1 - 2x - \alpha)w - 4x(1 - x) \geq 0,$$

where α is as in (2.4). We show that $g(x, a) \leq 0$ (for x in the appropriate range) and hence $w \geq a$ (which is (4.1)). Again we have two cases:

Case 1. $0 \leq a \leq (5 - t)/t$ *and* $\frac{1}{2} \geq x \geq 3a$.

Since $g(\frac{1}{2}, a) < 0$, it suffices to check that $g(3a, a) \leq 0$ or $a^2 - 2(1 - 6a - \alpha)a - 12a(1 - 3a) \leq 0$ or $49a \leq 14 - 2\alpha$. The latter holds for $0 \leq a \leq (5 - t)/t$ since it holds for $a = (5 - t)/t$ because $t > 245/(63 - 2\alpha) \approx 4.589$.

Case 2. $(5 - t)/t \leq a \leq 1/2t$ *and* $\frac{1}{2} \geq x \geq 4a - (5 - t)/t$.

As in Appendix 2 we have only to check the case $a = 1/2t$ and $x \geq 4a - (5 - t)/t = (t - 3)/t$. As in Case 1 we must only verify that $g((t - 3)/t, 1/2t) \leq 0$. But we chose $t = 121/(44 - 4\alpha)$ so that $g((t - 3)/t, 1/2t) = 0$.

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Note added in proof: Very recently, Lubotzky, Phillips, and Sarnak [LPS] constructed, for every prime p, a family of $\rho = p + 1$-regular graphs G with $\lambda_G \geq \rho - 2\sqrt{(\rho - 1)}$, where λ_G is as in Theorem 2.1. These graphs (choosing $p = 5$ or $p = 7$) together with Theorem 2.1, the construction described in Section 3, and the optimization of Appendix 2 lead to directed, acyclic superconcentrators of density less than 60. In a recent version of [Sh] (to appear in *Information and Control*) these graphs (choosing $p = 5$) and Theorem 2.1 yield nonacyclic s.c.-s of density 25 and undirected s.c.-s of density 13.

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